$\boxed{1.a}$ $y[n] + y[n-1] = x^2[n]$

Impulse response simply means the output when input is $\delta[n]$.

$y[n] + y[n-1] = \delta^2[n] = \delta[n]$ (because δ is either 1 or 0, and $1=1^2, 0=0^2$)

$$\xrightarrow{Z} \quad Y(z) + z^{-1}Y(z) = 1 \rightarrow Y(z) = \frac{1}{1+z^{-1}} = \frac{z}{z+1} \xrightarrow{Z^{-1}} \boxed{y[n] = (-1)^n u[n]}$$

This is clearly a nonlinear difference equation, so the impulse response isn't very useful.

$\boxed{1.b}$ $x[n] = \cos(\omega n)$

$$\cos^2(\omega n) = \frac{1}{2}(1 + \cos(2\omega n)) = \frac{1}{2} + \frac{1}{4}e^{j2\omega n} + \frac{1}{4}e^{-j2\omega n}$$

$$y[n] - y[n-1] = \frac{1}{2} + \frac{1}{4}e^{j2\omega n} + \frac{1}{4}e^{-j2\omega n} = x'[n]$$

Even though our equation is nonlinear, it is equivalent to the equation $y[n] - y[n-1] = x'[n]$, which is LTI. x' is a linear combination of eigensignals.

$$H(z) = \frac{1}{1+z^{-1}}, \quad H(e^{j\omega}) = \frac{1}{1+e^{-j\omega}}$$

$$y[n] = \frac{1}{2}H(e^{j0})\cdot 1 + \frac{1}{4}H(e^{j2\omega})e^{j2\omega n} + \frac{1}{4}H(e^{-j2\omega})e^{-j2\omega n}$$

$$= \boxed{\frac{1}{2}\frac{1}{1+1} + \frac{1}{4}\frac{1}{1+e^{-j2\omega}}e^{j2\omega n} + \frac{1}{4}\frac{1}{1+e^{+2j\omega}}e^{-j2\omega n}}$$

$$= \frac{1}{4} + \frac{1}{2}\left(\frac{1}{2}\frac{1}{1+e^{-2j\omega}}e^{j2\omega n} + \left(\frac{1}{2}\frac{1}{1+e^{+2j\omega}}e^{-j2\omega n}\right)^*\right)$$

$$= \frac{1}{4} + \mathrm{Re}\left(\frac{1}{2}\frac{1}{1+e^{-2j\omega}}e^{j2\omega n}\right) = \frac{1}{4} + \frac{1}{2}\mathrm{Re}\left(\frac{1+e^{+j2\omega}}{2+2\cos(2\omega)}e^{j2\omega n}\right)$$

$$= \boxed{\frac{1}{4} + \frac{1}{8\cos^2(\omega)}\left(\cos(2\omega n) + \cos(2\omega(n+1))\right)} \quad \omega \neq \frac{\pi}{2} + \pi k, \; k = 0, \pm 1, \pm 2, \ldots$$

1.b cont

Special case: $\omega = \frac{\pi}{2} + \pi k$. These can be reduced to just $\omega = \frac{\pi}{2}$.

$$\cos^2(\omega n) = \cos^2\left(\left(\frac{\pi}{2} + \pi k\right)n\right) = \cos^2\left(\frac{\pi}{2}n + \pi k n\right) = \begin{cases} \cos^2\left(\frac{\pi}{2}n\right) & \text{if } kn \text{ is even} \\ \left(-\cos\left(\frac{\pi}{2}n\right)\right)^2 & \text{if } kn \text{ is odd} \end{cases}$$

$$= \cos^2\left(\frac{\pi}{2}n\right)$$

$$x'[n] = \frac{1}{2}\left(1 + \cos(\pi n)\right) = \frac{1}{2}\left(1 + (-1)^n\right).$$

I know I haven't really talked about this, but is a very important special case. The eigenvalue blows up, because it corresponds to a pole of the system.

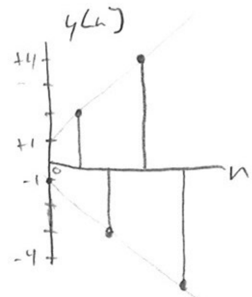
$$H(e^{j\pi}) = \frac{1}{1 + e^{-j\pi}} = \frac{1}{1-1} = \text{undef.}$$

If $x'[n] = (-1)^n u[n]$, then it would simply grow linearly.

$$y[n] = \mathcal{Z}^{-1}\left\{\left(\frac{z}{z+1}\right)^2\right\} = \mathcal{Z}^{-1}\left\{\frac{z}{z+1} - \frac{z}{(z+1)^2}\right\} = \left((-1)^n + n(-1)^n\right)u[n]$$

$$= (n+1)(-1)^n u[n]$$

Thus it is not BIBO stable. This is resonance. The system is "tuned" for this input, and amplifies it. This input isn't really an eigensignal, though in a sense it is if we take the eigenvalue to be infinite.



γ^n is still an eigensignal for all LTI systems for all $\gamma \in \mathbb{C}$. $H(\gamma)$ is still the eigenvalue. But whenever there is an infinity (in this case, in $H(\gamma)$), all bets are off and we have to look deeper.

2.)

$$y[n] - \alpha y[n-1] = x[n] = 2u[n]$$

$$\xrightarrow{z} \quad Y(z) - \alpha z^{-1} Y(z) = \frac{2z}{z-1}$$

$$Y(z) = \frac{1}{1-\alpha z^{-1}} \frac{2z}{z-1} = \frac{2z^2}{(z-\alpha)(z-1)}$$

$$\frac{Y(z)}{z} = \frac{2z}{(z-\alpha)(z-1)} = \frac{A}{z-\alpha} + \frac{B}{z-1}$$

$$A = \lim_{z\to\alpha} \frac{Y(z)}{z}(z-\alpha) = \lim_{z\to\alpha} \frac{2z}{z-1} = \frac{2\alpha}{\alpha-1}$$

$$B = \lim_{z\to 1} \frac{Y(z)}{z}(z-1) = \lim_{z\to 1} \frac{2z}{z-\alpha} = \frac{2}{1-\alpha} = \frac{-2}{\alpha-1}$$

$$Y(z) = \frac{2}{\alpha-1}\left(\alpha \frac{z}{z-\alpha} + \frac{-z}{z-1}\right)$$

$$y[n] = \frac{2}{\alpha-1}\left(\alpha\, \alpha^n - 1\right) u[n]$$

$$y[1] = \frac{2}{\alpha-1}\left(\alpha\, \alpha^1 - 1\right) \geq 1$$

$$2(\alpha^2-1) = 2(\alpha+1)(\alpha-1) \geq (\alpha-1)$$

$2(\alpha+1) \geq 1$ (this is OK, we are excluding $\alpha = 1$)

$$\alpha \geq \tfrac{1}{2} - 1$$

$\alpha \geq -\frac{1}{2}$ so $\boxed{\text{any } \alpha \in (0,1) \text{ will work}}$

A simpler way is to just compute $y[1]$:

$y[-1]=0$, $y[0] = \alpha y[-1] + x[0] = 2$, $y[1] = \alpha y[0] + x[1] = 2\alpha + 2 = 2(\alpha+1)$

$y[1] = 2(\alpha+1) \geq 1 \rightarrow \alpha \geq \frac{1}{2} - 1 = -\frac{1}{2}$

3.a)

$$y[n+2] + \tfrac{1}{2} y[n+1] - \tfrac{1}{2} y[n] = x[n+1] + x[n]$$

$$\xrightarrow{\mathcal{Z}} \quad z^2 Y(z) + \tfrac{1}{2} z Y(z) - \tfrac{1}{2} Y(z) = z X(z) + X(z)$$

$$\left(z^2 + \tfrac{1}{2} z - \tfrac{1}{2}\right) Y(z) = (z+1)(z - \tfrac{1}{2}) Y(z) = (z+1) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z+1}{(z+1)(z-\frac{1}{2})} = \boxed{\frac{1}{z-\frac{1}{2}}}$$

$$\boxed{h[n] = 2^{-(n-1)} u[n-1]}$$

3.b)

$h[n]$ is causal, so the system is causal.

Also, $y[n] = -\frac{1}{2} y[n-1] + \frac{1}{2} y[n-2] + x[n-1] + x[n-2]$

present output does not depend on future inputs or outputs, thus it is causal.

$h[n]$ is absolutely summable: $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=1}^{\infty} 2^{-n} = \frac{(\frac{1}{2})^1 - (\frac{1}{2})^{\infty+1}}{\frac{1}{2} - 1} = \frac{\frac{1}{2}}{\frac{1}{2}-1} = \frac{1}{1-2} = -$

So it is BIBO Stable.

Also, all poles are inside the unit circle.

3.c)

$x[n] = 4^{-n} u[n], \quad y[-1] = y[-2] = 1$

$$X(z) = \frac{z}{z - \frac{1}{4}}$$

$$Y_p(z) = H(z) X(z) = \frac{1}{z-\frac{1}{2}} \frac{z}{z-\frac{1}{4}} = 4\left(\frac{z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{4}}\right)$$

$$\frac{Y_p(z)}{z} = \frac{1}{(z-\frac{1}{2})(z-\frac{1}{4})} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-\frac{1}{4}}$$

$$y_p[n] = 4\left(2^{-n} - 4^{-n}\right) u[n]$$

$$A = \lim_{z \to \frac{1}{2}} \frac{Y_p(z)}{z} (z - \tfrac{1}{2}) = \frac{1}{\frac{1}{2} - \frac{1}{4}} = \frac{1}{(\frac{1}{4})} = 4$$

$$B = \lim_{z \to \frac{1}{4}} \frac{Y_p(z)}{z} (z - \tfrac{1}{4}) = \frac{1}{\frac{1}{4} - \frac{1}{2}} = \frac{1}{(-\frac{1}{4})} = -4$$

3.c cont.

$$y[n+2] + \frac{1}{2}y[n+1] - \frac{1}{2}y[n] = 0$$

$$\left(z^2 + \frac{1}{2}z - \frac{1}{2}\right) = 0 \rightarrow z = \left\{-1, \frac{1}{2}\right\}$$

$$y_0[n] = a(-1)^n + b\,2^{-n}$$

$$y[-1] = 1 = -a + 2b + \underbrace{y_p[-1]}_{=0} \rightarrow -a + 2b = 1$$

$$y[-2] = 1 = +a + 4b + \underbrace{y_p[-2]}_{=0} \rightarrow a + 4b = 1$$

$$\left[\begin{array}{cc|c} 1 & 4 & 1 \\ -1 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc} 1 & 4 & 1 \\ 0 & 6 & 2 \end{array}\right]$$

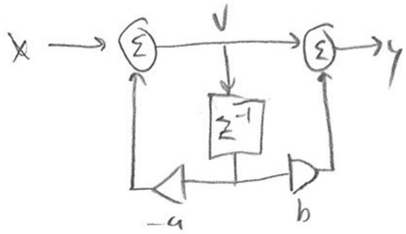
$$6b = 2 \rightarrow b = \frac{2}{6} = \frac{1}{3}$$

$$-a + 2\frac{1}{3} = 1 \rightarrow -a = 1 - \frac{2}{3} = \frac{1}{3} \rightarrow a = -\frac{1}{3}$$

$$y_0[n] = \frac{1}{3}\left(2^{-n} - (-1)^n\right)$$

$$\boxed{y[n] = \left(\frac{1}{3} + 4\right)2^{-n} - \frac{1}{3}(-1)^n - 4^{-n} \quad , n \geq 0}$$

4



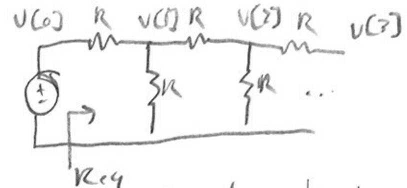
Z-domain:

$$V = X + -az^{-1}V \rightarrow V(1 + az^{-1}) = X$$

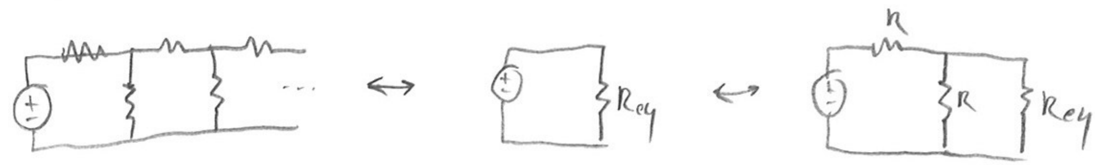
$$Y = V + bz^{-1}V \rightarrow Y = (1 + bz^{-1})V$$

$$H = \frac{Y}{X} = \frac{(1+bz^{-1})V}{(1+az^{-1})V} = \boxed{\frac{z+b}{z+a}}$$

5.



There is a trick that is easier than solving a difference equation:
The ladder network is infinitely long, so it doesn't matter if we add another "rung" to it.



So, $R \,||\, R_{eq} + R = R_{eq} \rightarrow \dfrac{R R_{eq}}{R + R_{eq}} + R = R_{eq}$

$$R R_{eq} + R(R + R_{eq}) = R_{eq}(R + R_{eq})$$

$$R R_{eq} + R^2 + \cancel{R R_{eq}} = \cancel{R_{eq} R} + R_{eq}^2$$

$$R_{eq}^2 - R R_{eq} - R^2 = 0$$

$$R_{eq} = \frac{+R \pm \sqrt{R^2 - 4(-R^2)(1)}}{2} = \frac{R \pm \sqrt{R^2(1+4)}}{2} = R\left(\frac{1 \pm \sqrt{5}}{2}\right)$$

$\frac{1-\sqrt{5}}{2} < 0$, so $\boxed{R_{eq} = R \, \frac{1+\sqrt{5}}{2}}$

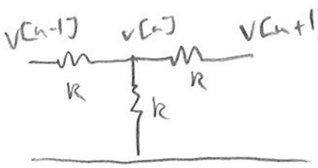
$\frac{1+\sqrt{5}}{2}$ is called the Golden Ratio, and typically denoted as ϕ. It often appears in problems involving self-similarity such as this, and is a very interesting number in and of itself.

$$\phi^2 = 1 + \phi \rightarrow \phi = \sqrt{1+\phi} = \sqrt{1 + \sqrt{1+\phi}} = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}$$

$$\phi^{-1} = \phi - 1 \rightarrow \phi = \frac{1}{1-\phi} = \frac{1}{1 - \frac{1}{1-\phi}} = \frac{1}{1 - \frac{1}{1 - \frac{1}{1 - \cdots}}}$$

5. cont

The trick is easy once you know it, but isn't very general and requires being clever. Now we solve it with a difference equation:



node equation at node n. $(n \geq 1)$:

$$\frac{V[n] - V[n-1]}{R} + \frac{V[n] - V[n+1]}{R} + \frac{V[n]}{R} = 0$$

$$3V[n] - V[n-1] - V[n+1] = 0$$

$$V[n+2] - 3V[n+1] + V[n] = 0$$

Characteristic modes are:

$$z^2 - 3z + 1 = 0 \rightarrow z = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2} = \{\gamma_+, \gamma_-\}$$

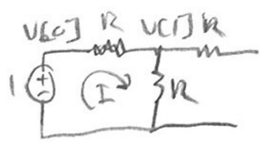
$\sqrt{4} < \sqrt{5} < \sqrt{9} \rightarrow 2 < \sqrt{5} < 3$, so $\gamma_- = \frac{3-\sqrt{5}}{2}$ is $0 < \gamma_- < 1$

$\gamma_+ = \frac{3+\sqrt{5}}{2}$ is obviously > 1 because $\frac{3}{2} > 1$ and $\frac{\sqrt{5}}{2} > 0$

$V[n] = a\gamma_-^n + b\gamma_+^n$ b must be zero, because the voltage must get smaller as n increases (it's a bunch of voltage dividers)

a is whatever the source is. We can set it to 1 without loss of generality.

$$V[n] = \gamma_-^n = \left(\frac{3-\sqrt{5}}{2}\right)^n$$



$$I = \frac{V[0] - V[1]}{R} = \frac{1 - \gamma_-}{R}$$

$$R_{eq} = \frac{V[0]}{I} = \frac{R}{1-\gamma_-} = R\,\frac{1}{1 - \frac{3}{2} + \frac{\sqrt{5}}{2}} = R\,\frac{1}{\frac{-1+\sqrt{5}}{2}} = \frac{2R}{\sqrt{5}-1}$$

$$\frac{2}{\sqrt{5}-1} = \frac{2(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)} = \frac{2(\sqrt{5}+1)}{5-1} = \frac{1+\sqrt{5}}{2} = \phi \qquad \boxed{R_{eq} = \phi R}$$